Day 4: Logic and Topology for Knowledge, Knowability, and Belief

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Overview

What is the relationship between belief, evidence, and knowledge?

This is a very old question, but modern work in epistemic logic offers new approaches and insights.

In fact, *topological* models provide a framework naturally suited to the representation of *evidence* and its relationship to knowledge and belief.

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Our starting point is a proposal of Stalnaker's for a logic of knowledge and belief.

- We refine and extend this proposal by carefully distinguishing two epistemic notions:
  (1) an "evidence-in-hand" conception of knowledge, and
  (2) a weaker "evidence-out-there" notion of what *could come to be known*.
- To do this, we import Stalnaker's principles into a richer semantic setting based on *topological subset spaces*.
  - These models are rich enough to respect the distinction between (1) and (2), yielding a trimodal logic of knowledge, knowability, and belief.

Logic for knowledge and belief

Let $\mathcal{L}_{K,B}$ denote a classical propositional language augmented with modalities K and B for *knowledge* and *belief*:

$$\varphi ::= p \,|\, \neg\varphi \,|\, \varphi \wedge \psi \,|\, \varphi \vee \psi \,|\, \varphi \to \psi \,|\, K\varphi \,|\, B\varphi.$$

So the formula $K\varphi$ expresses knowledge of φ, while $B\varphi$ expresses belief in φ.

Using this language, one can articulate a variety of postulates about knowledge, belief, and their interplay.

Logic for knowledge and belief

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$$K\varphi \to \varphi$$

"If you know φ, then φ is true."

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Logic for knowledge and belief

Which assumptions are appropriate? That depends on the particular conception of knowledge/belief one seeks to model.

- Modal logic does not and cannot arbitrate the "ultimate truth" of such assumptions.
- Rather, it provides a framework for reasoning formally about the relationships between different assumptions and their logical consequences.

Stalnaker's system

Stalnaker has proposed a logic intended to capture the relationship between knowledge and belief, where belief is interpreted in the strong sense of *subjective certainty*.

This logic extends the classical S4 system for knowledge...

(K_K)	$K(\varphi \to \psi) \to (K\varphi \to K\psi)$	Distribution
(T_K)	$K\varphi \to \varphi$	Factivity
(4_K)	$K\varphi \to KK\varphi$	Positive introspection
(Nec_K)	from φ infer $K\varphi$	Necessitation

$S4_K$ axioms for knowledge

Stalnaker's system

...with the following additional axioms.

(D_B)	$B\varphi \to \neg B \neg \varphi$	Consistency of belief
(sPI)	$B\varphi \to KB\varphi$	Strong positive introspection
(sNI)	$\neg B\varphi \to K\neg B\varphi$	Strong negative introspection
(KB)	$K\varphi \to B\varphi$	Knowledge implies belief
(FB)	$B\varphi \to BK\varphi$	Full belief

Stalnaker's additional axioms

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- Knowledge is belief plus something extra.

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"What you know you also believe."

- Captures the idea that knowledge is stronger than belief.
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$$B\varphi \to BK\varphi$$

"If you believe φ, then you believe that you know φ."

- Belief is not subjectively distinguishable from knowledge.
- This captures the "strong" sense of belief Stalnaker is after.
  - An agent who feels certain that φ is true also feels certain that she *knows* that φ is true.

Stalnaker's system

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$$B\varphi \leftrightarrow \hat{K}K\varphi,$$

where $\hat{K}$ abbreviates $\neg K \neg$.

- This says that belief is equivalent to "the epistemic possibility of knowledge".
- In particular, in this system belief can be *defined* in terms of knowledge.

Stalnaker's system

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Given the strong sense of belief Stalnaker seeks to capture, each of (KB) and (FB) has a certain plausibility.

However, their *joint* plausibility starts to waver when we push on just what *knowledge* is supposed to mean.

Somewhat more precisely: tension between (KB) and (FB) emerges when knowledge is interpreted more concretely in terms of what is justified by a body of evidence.

Knowledge from evidence

Consider the following informal account: an agent *knows* something just in case it is entailed by the available evidence.

- E.g., in a card game one player might be said to *know* their opponent is not holding two aces on the basis of the fact that they are themselves holding three aces.

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The standard *possible worlds, relational semantics* for epistemic logic can be viewed as a formalization of this intuition.

- Roughly: each world w is associated with a set of possible worlds $R(w)$; the agent is said to *know* φ *at* w just in case φ is true at all worlds in $R(w)$.
- Think of the worlds in $R(w)$ as those compatible with the evidence at w; then the agent knows φ just in case the evidence rules out all not-φ possibilities.

Evidence “in hand”

Another simple example: you’ve measured your height to be 5 feet 10 inches, ± 1 inch. With this measurement in hand, you might be said to *know* that you are less than 6 feet tall (having ruled out the possibility that you are taller).

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- If you have evidence-in-hand that entails φ, you should be certain of φ.

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- If you have evidence-in-hand that entails φ, you should be certain of φ.

It does *not* sit comfortably with (FB) ($B\varphi \to BK\varphi$).

- The implication seems false: you can be (subjectively) certain of φ ($B\varphi$) without also being certain that you currently have evidence-in-hand that guarantees φ ($BK\varphi$).

Evidence “out there”

Consider now a weaker, *existential* interpretation of “available evidence”: *there is* evidence (somewhere out there) entailing φ.

- Not necessarily “in hand” at the moment.
- Intuitively, we’ve shifted from what’s *known* to what’s *knowable*.

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- If you are certain of φ, then you are certain that there is evidence entailing φ.
- Only believe what you think you could come to know.

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- If you are certain of φ, then you are certain that there is evidence entailing φ.
- Only believe what you think you could come to know.

(KB) $(K\varphi \rightarrow B\varphi)$ falters.

- The mere fact that you could, in principle, discover evidence entailing φ should not in itself imply that you believe φ.

Knowledge and knowability

It seems we want the "evidence-in-hand" intuition for (KB), and the "evidence-out-there" intuition for (FB). Let's take both.

Let $\mathcal{L}_{K,\Box,B}$ denote the language $\mathcal{L}_{K,B}$ extended with a new unary modality $\Box$.

- Write $K\varphi$ for "φ is entailed by the evidence-in-hand".
  - Gloss: "φ is known".
- Write $\Box\varphi$ for "φ is entailed by the evidence-out-there".
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(FB) becomes (RB), "responsible belief":

$$(B\varphi \rightarrow BK\varphi) \rightsquigarrow (B\varphi \rightarrow B\Box\varphi).$$

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Recall that the *interior* operator naturally corresponds to a notion of knowability:

$$x \in int(A) \quad \Leftrightarrow \quad \exists U \in \mathcal{T}(x \in U \subseteq A).$$

The interior of A consists of all those points x for which there is a feasible measurement U (i.e., $U \in \mathcal{T}$ and $x \in U$) that entails A (i.e., $U \subseteq A$).

Topological semantics

Consider the language generated by:

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Formulas of this language are interpreted in **topological subset models** $M = (X, \mathcal{T}, v)$ with respect to pairs of the form (x, U) where $x \in U \in \mathcal{T}$.

- Such pairs are called *epistemic scenarios*: x represents the actual world, and U represents the agent's evidence-in-hand.

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$$\begin{array}{lll}
(x,U) \models p & \text{iff} & x \in v(p) \\
(x,U) \models \neg\varphi & \text{iff} & (x,U) \not\models \varphi \\
(x,U) \models \varphi \wedge \psi & \text{iff} & (x,U) \models \varphi \text{ and } (x,U) \models \psi \\
(x,U) \models K\varphi & \text{iff} & U \subseteq [\![\varphi]\!]^U \\
(x,U) \models \Box\varphi & \text{iff} & x \in int([\![\varphi]\!]^U),
\end{array}$$

where $[\![\varphi]\!]^U = \{x \in U \ : \ (x,U) \models \varphi\}$.

Logic for knowledge and knowability

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- S5 $=$ S4 $+ (\neg K\varphi \to K\neg K\varphi)$, the "negative introspection" axiom.

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- $\mathsf{S5} = \mathsf{S4} + (\neg K\varphi \to K\neg K\varphi)$, the "negative introspection" axiom.

Theorem

The language $\mathcal{L}_{K,\Box}$ interpreted as above is axiomatized by

$$\mathsf{EL}_{K,\Box} = \mathsf{S5}_K + \mathsf{S4}_\Box + (K\varphi \to \Box\varphi).$$

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This will serve as our "basic logic of knowledge and knowability" (analogous to $\mathsf{S4}_K$ in Stalnaker's system).

Logic for knowledge, knowability, and belief

What about belief?

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- ⋆ This is noteworthy: once we carefully distinguish knowledge from knowability, Stalnaker's postulates no longer imply that belief is reducible.

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Let $\mathsf{SEL}_{K,\Box,B}$ denote $\mathsf{EL}_{K,\Box}$ together with the following:

(K_B)	$B(\varphi \to \psi) \to (B\varphi \to B\psi)$	Distribution of belief
(sPI)	$B\varphi \to KB\varphi$	Strong pos. introspection
(KB)	$K\varphi \to B\varphi$	Knowledge implies belief
(RB)	$B\varphi \to B\Box\varphi$	Responsible belief
(wF)	$B\varphi \to \Diamond\varphi$	Weak factivity
(CB)	$B(\neg\Box\varphi \to \Box\neg\Box\varphi)$	Confident belief

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(K_B), (sPI), and (KB) are theorems of Stalnaker's original system.
(RB) is the translation of (FB) we have already discussed.

Logic for knowledge, knowability, and belief

Both (wF) and (CB) become theorems of Stalnaker's original system if we "forget" the distinction between $\Box$ and K

- I.e., replace every $\Box$ with K (and every $\Diamond$ with $\hat{K}$).

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Weak factivity:

$$B\varphi \to \Diamond\varphi$$

"If you are certain of φ, then φ cannot be knowably false."

- Weaker form of factivity ($B\varphi \to \varphi$).
- You can believe false things, but you can't believe *knowably* false things.

Logic for knowledge, knowability, and belief

Confident belief:

$$B(\neg\Box\varphi \to \Box\neg\Box\varphi)$$

"You believe that if φ is unknowable, it is knowably unknowable."

- Faith in the justificatory power of evidence.
- You are sure that φ is either knowable or, if not, that you could come to know that it is unknowable.
  - Topologically: "no boundary cases", i.e., cases where no measurement can entail φ yet every measurement leaves open the possibility that some further measurement will.

Logic for knowledge, knowability, and belief

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$\mathsf{SEL}_{K,\Box,B}$ proves the following equivalence:

$$B\varphi \leftrightarrow K\Diamond\Box\varphi.$$

- Belief is definable from knowledge *and knowability*.
- To believe φ is to know that every measurement you might take leaves open the possibility of taking some further measurement that would guarantee φ.

Logic for knowledge, knowability, and belief

In fact, this equivalence characterizes $\mathsf{SEL}_{K,\Box,B}$ as an extension of $\mathsf{EL}_{K,\Box}$ in the following sense:

Proposition

$\mathsf{EL}_{K,\Box} + (B\varphi \leftrightarrow K\Diamond\Box\varphi)$ *and* $\mathsf{SEL}_{K,\Box,B}$ *prove the same theorems.*

Logic for knowledge, knowability, and belief

Semantically, this equivalence corresponds to a particularly appealing topological interpretation of belief:

$$\begin{aligned}(x,U) \models B\varphi \quad &\text{iff} \quad (x,U) \models K\Diamond\Box\varphi \\ &\text{iff} \quad U \subseteq cl(int(\llbracket\varphi\rrbracket^U)) \\ &\text{iff} \quad \llbracket\varphi\rrbracket^U \text{ has dense interior in } U.\end{aligned}$$

- Dense interior is a standard topological notion of "largeness".
  - These are precisely the sets with *nowhere dense* complements.

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Intuitively, such a set fills "almost all" of the space. Morally, then:

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- Knowledge = truth in *all* possible alternatives.
- Belief = truth in *almost all* possible alternatives.

Logic for knowledge, knowability, and belief

Theorem

$\mathsf{SEL}_{K,\Box,B}$ is a sound and complete axiomatization of $\mathcal{L}_{K,\Box,B}$ with respect to the class of topological subset models using the semantics for belief just presented.

Theorem

$\mathsf{SEL}_{K,\Box,B}$ proves all the $\mathsf{KD45}$ principles for belief. In fact, $\mathsf{KD45}_B$ is a sound and complete axiomatization of the fragment $\mathcal{L}_B$ with respect to the class of topological subset models.

Weaker notions of belief

We adopted weak factivity (wF) and confident belief (CB) in order to obtain a reduction result for belief analogous to Stalnaker's.

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- We do this by introducing the *doxastic range*—intuitively, collecting the "most plausible" worlds compatible with the evidence-in-hand.

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- We do this by introducing the *doxastic range*—intuitively, collecting the "most plausible" worlds compatible with the evidence-in-hand.
- We also import the idea of topological "almost all" quantification directly into the semantics to produce models for (CB) without (wF).

Weaker notions of belief

Let $\mathsf{EL}_{K,\Box,B}$ be the logic obtained by dropping the axioms (wF) and (CB) from $\mathsf{SEL}_{K,\Box,B}$.

As before, we rely on topological subset models; however, we now define the evaluation of formulas with respect to *epistemic-doxastic (e-d) scenarios*, which are tuples of the form (x, U, V) where (x, U) is an epistemic scenario, $V \in \mathcal{T}$, and $V \subseteq U$.

- Call V the *doxastic range*.

Weaker notions of belief

The key semantic clauses are:

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\begin{array}{lll}
(x, U, V) \models K\varphi & \text{iff} & U = [\![\varphi]\!]^{U,V} \\
(x, U, V) \models \Box\varphi & \text{iff} & x \in int([\![\varphi]\!]^{U,V}) \\
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- Modalities K and $\Box$ are interpreted (essentially) as before.

Weaker notions of belief

The key semantic clauses are:

$$\begin{array}{lll} (x,U,V) \models K\varphi & \text{iff} & U = [\![\varphi]\!]^{U,V} \\ (x,U,V) \models \Box\varphi & \text{iff} & x \in int([\![\varphi]\!]^{U,V}) \\ (x,U,V) \models B\varphi & \text{iff} & V \subseteq [\![\varphi]\!]^{U,V}, \end{array}$$

where

$$[\![\varphi]\!]^{U,V} = \{x \in U \,:\, (x,U,V) \models \varphi\}.$$

- Modalities K and $\Box$ are interpreted (essentially) as before.
- Belief is universal quantification over the doxastic range. Intuitively:
  - V is the agent's "conjecture" about the world, typically stronger than what is guaranteed by her evidence-in-hand U.
  - States in V are considered "more plausible" than the other states in U, so belief = truth in all these more plausible states.

Weaker notions of belief

Note that we do not require that $x \in V$; this corresponds to the intuition that the agent may have false beliefs.

In order to distinguish these semantics from those previous, we refer to them as *epistemic-doxastic (e-d) semantics* for topological subset spaces.

Weaker notions of belief

Theorem

$\mathsf{EL}_{K,\Box,B}$ *is a sound and complete axiomatization of* $\mathcal{L}_{K,\Box,B}$ *with respect to the class of all topological subset spaces under e-d semantics.*

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Theorem

$\mathsf{EL}_{K,\Box,B} + (\mathsf{wF})$ *is a sound and complete axiomatization of* $\mathcal{L}_{K,\Box,B}$ *with respect to the class of all topological subset spaces under e-d semantics for dense e-d scenarios.*

Weaker notions of belief

The reductive interpretation of the belief modality, namely

$$(x, U) \models B\varphi \quad \text{iff} \quad U \subseteq cl(int([\![\varphi]\!]^U)),$$

does *not* arise as a special case of our new e-d semantics.

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- There is no condition (like density) one can put on the doxastic range V that recovers this interpretation of B.
- Roughly: formulas of the form $\neg\Box\varphi \to \Box\neg\Box\varphi$ correspond to the open and dense sets, and in general no (nonempty) open set V is contained in *every* open, dense set.

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- Roughly: formulas of the form $\neg\Box\varphi \to \Box\neg\Box\varphi$ correspond to the open and dense sets, and in general no (nonempty) open set V is contained in *every* open, dense set.
- Upshot: we can't hope to validate (CB) in the e-d semantics just presented without also validating $B\bot$.

Weaker notions of belief

Solution: alter the semantic interpretation of the belief modality so it "ignores" nowhere dense sets:

$$
\begin{array}{lll}
(x,U,V) \mathrel{|\!\approx} p & \text{iff} & x \in v(p) \\
(x,U,V) \mathrel{|\!\approx} \neg\varphi & \text{iff} & (x,U,V) \mathrel{|\!\not\approx} \varphi \\
(x,U,V) \mathrel{|\!\approx} \varphi \wedge \psi & \text{iff} & (x,U,V) \mathrel{|\!\approx} \varphi \text{ and } (x,U,V) \mathrel{|\!\approx} \psi \\
(x,U,V) \mathrel{|\!\approx} K\varphi & \text{iff} & U = [\![\varphi]\!]^{U,V} \\
(x,U,V) \mathrel{|\!\approx} \Box\varphi & \text{iff} & x \in int([\![\varphi]\!]^{U,V}) \\
(x,U,V) \mathrel{|\!\approx} B\varphi & \text{iff} & V \subseteq^* [\![\varphi]\!]^{U,V},
\end{array}
$$

where

$$[\![\varphi]\!]^{U,V} = \{x \in U : (x,U,V) \mathrel{|\!\approx} \varphi\},$$

and we write $A \subseteq^* B$ iff $A - B$ is nowhere dense.

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\end{array}
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where

$$[\![\varphi]\!]^{U,V} = \{x \in U \ : \ (x,U,V) \approx \varphi\},$$

and we write $A \subseteq^* B$ iff $A - B$ is nowhere dense.

The belief modality now effectively quantifies over *almost all* worlds in the doxastic range V rather than over all worlds.

Weaker notions of belief

Theorem

$\mathsf{EL}_{K,\Box,B} + (\mathsf{CB})$ *is a sound and complete axiomatization of* $\mathcal{L}_{K,\Box,B}$ *with respect to the class of all topological subset spaces under e-d semantics using the semantics given above:*

$$\forall \varphi \in \mathcal{L}_{K,\Box,B} (\approx \varphi \Leftrightarrow \vdash_{\mathsf{EL}_{K,\Box,B}+(\mathsf{CB})} \varphi).$$

Thank you!