Days 2 & 3: Topology & Topological Semantics

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Adam Bjorndahl

Carnegie Mellon University

Space?

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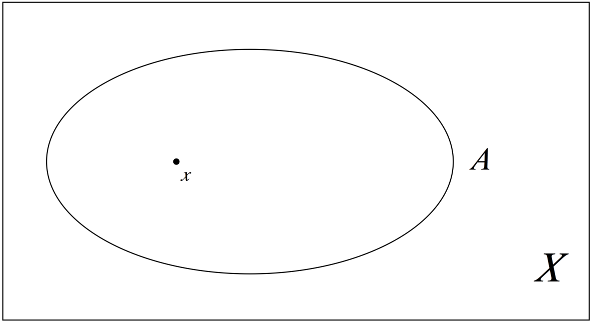
What could this have to do with epistemology?

Robustness



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Set membership is a binary affair.

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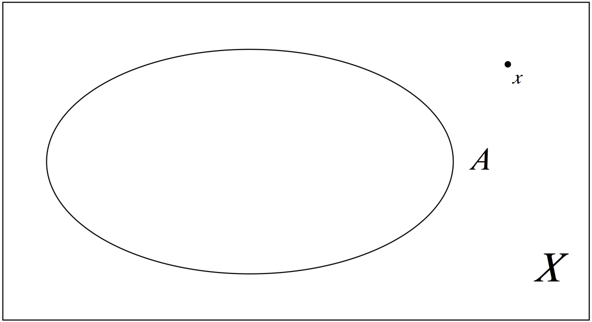

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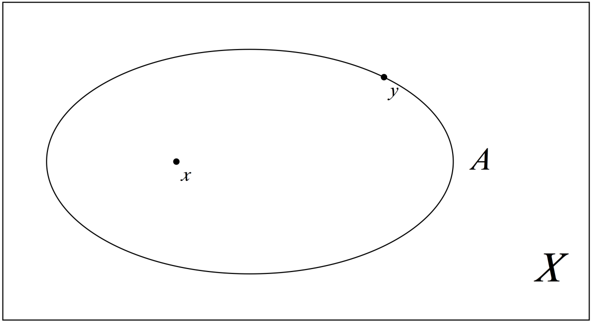

Does it ever make sense to think of membership as a graded notion? Spatial intuitions provide a context where this seems natural: the point x is “fully” or “robustly” in the set A, whereas y is only just “barely” in A.

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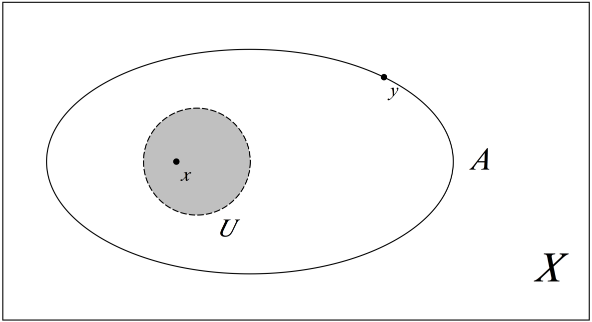

How can this intuition be formalized?

Idea: x is *robustly* in A if there exists a set U such that $x \in U$ and $U \subseteq A$. U acts as a "witness" to x's membership in A.

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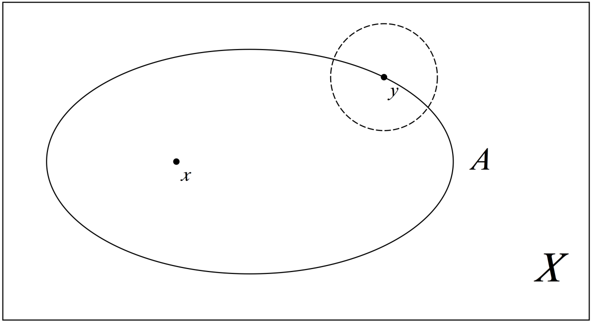

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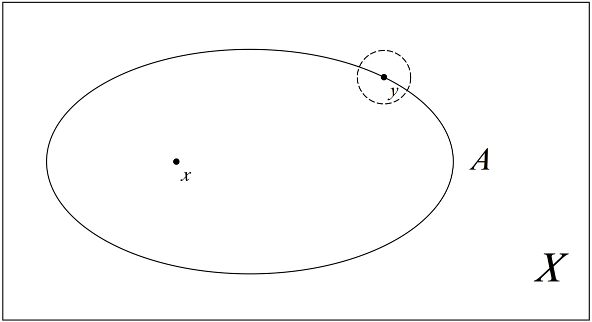

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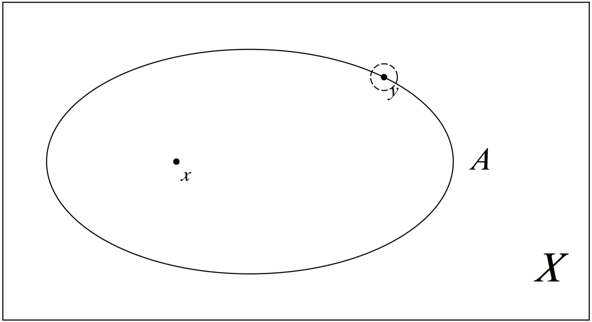

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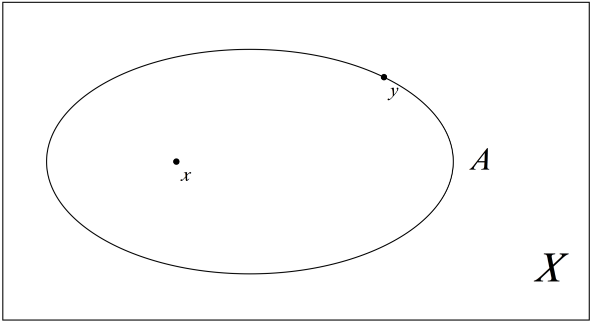

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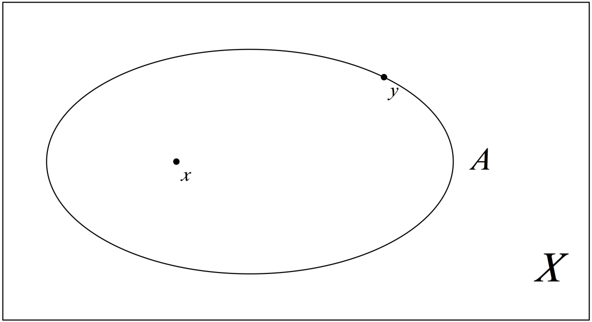

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Solution: restrict what counts as a “witness”.

Topological spaces

A **topological space** is a set X together with a collection $\mathcal{T} \subseteq 2^X$ of *open sets* (satisfying certain constraints).

- $\mathcal{T}$ is called a *topology* on X.

Open sets are what count as "possible witnesses". We say that x is in the **interior** of A, and write $x \in int(A)$, if there exists an open set $U \in \mathcal{T}$ such that $x \in U \subseteq A$.

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- Intuition: each open set U represents a certain notion of "nearness".
- A point x is in the interior of a set A iff all "nearby" points (according to some notion of nearness) are also in A.

Topological spaces

Officially, to be a topology on X, $\mathcal{T}$ must satisfy the following:

- $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$.
- If $U, V \in \mathcal{T}$ then also $U \cap V \in \mathcal{T}$.
  - "$\mathcal{T}$ is closed under pairwise intersections."
- If $\mathscr{C} \subseteq \mathcal{T}$ then also $\bigcup \mathscr{C} \in \mathcal{T}$.
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In practice, one often specifies a topology by identifying a set of "basic" open sets and then simply throwing in all their unions.

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- The open interval of radius ε centered at $c \in \mathbb{R}$ is:

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- According to this topological structure, $x \in int(A)$ just in case every real number "sufficiently close" to x (i.e., within some ε) is also in A.
- This can be generalized to higher-dimensional Euclidean spaces $\mathbb{R}^n$ by interpreting "radius ε" using the appropriate n-dimensional notion of distance.
  - Think: Pythagoras.

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- $\mathcal{T} = 2^X$, the full powerset of X.
  - Every set is open, including every singleton set.
  - Intuitively: nothing is "close" to anything because every point can be individually separated with an open set.
  - Formally, for all $A \subseteq X$, we have $x \in int(A)$ iff $x \in A$.
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- $\mathcal{T} = \{\emptyset, X\}$, the smallest possible topology on X.
  - Intuitively: everything is "close" to everything, because no point can be separated from anything else with an open set.
  - Formally, for all $A \subseteq X$, $int(A) = \emptyset$ unless $A = X$.
  - This is called the *indiscrete topology* on X.

The interior operator

It is easy to see that

$$\begin{aligned} int(A) &= \{x \in X \ : \ (\exists U \in \mathcal{T})(x \in U \subseteq A)\} \\ &\subseteq A, \end{aligned}$$

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The interior operator also satisfies several other properties, each of which can be proved fairly straightforwardly (these are good exercises!):

- $int(X) = X$
- $int(A \cap B) = int(A) \cap int(B)$
- $int(int(A)) = int(A)$
- If $A \subseteq B$ then $int(A) \subseteq int(B)$

Closure

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- Intuition: being almost in A is the same as *not* being robustly in the complement of A. Proof: exercise.

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In fact, given a set X together with an operator $I : 2^X \to 2^X$ satisfying the properties from the previous slide...

- $I(X) = X$, $I(A \cap B) = I(A) \cap I(B)$, $I(A) \subseteq A$, $I(I(A)) = I(A)$, $A \subseteq B \Rightarrow I(A) \subseteq I(B)$

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...we can define a topology by setting

$$\mathcal{T}_I = \{A \subseteq X \ : \ I(A) = A\},$$

and the interior operator corresponding to this topology coincides exactly with I.

Topological semantics

Consider the basic modal language generated by

$$\varphi ::= p \,|\, \neg\varphi \,|\, \varphi \wedge \psi \,|\, \Box\varphi,$$

where $p \in \text{PROP}$.

Formulas of this language can be interpreted in **topological models** $M = (X, \mathcal{T}, v)$, where:

- $(X, \mathcal{T})$ is a topological space, and
- $v : \text{PROP} \to 2^X$ is a valuation.

The truth set of φ is defined as before, with one major difference:

$$[\![\Box\varphi]\!] \quad = \quad int([\![\varphi]\!]).$$

The "logic of space"

Topology	Modal Logic
$int(X) = X$	$\Box\top \leftrightarrow \top$
$int(A \cap B) = int(A) \cap int(B)$	$\Box(\varphi \wedge \psi) \leftrightarrow (\Box\varphi \wedge \Box\psi)$
$int(A) \subseteq A$	$\Box\varphi \to \varphi$
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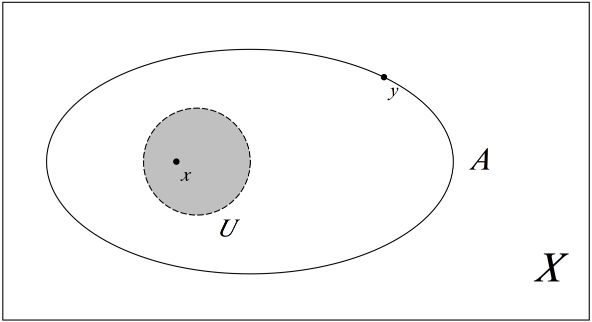
- Every statement one can articulate in this language that is valid in all topological spaces is derivable from these principles (plus classical reasoning).

Topology as epistemology



Think of U as a piece of evidence that (imperfectly) indicates the true state of the world: the points in U are precisely those that are compatible with the evidence.

E.g., U might be the result of some measurement with error.

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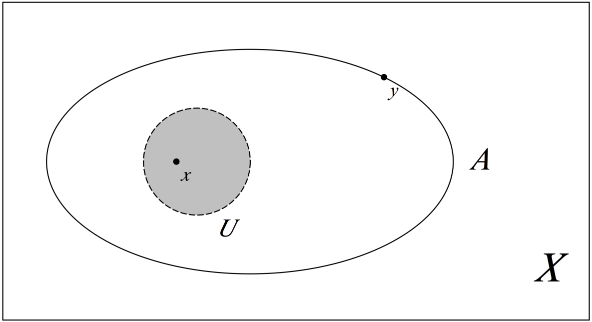

Such a "measurement" U is not precise enough to tell you the exact state of the world.

However, it can still be informative: in the above, it *is* precise enough to indicate that A holds.

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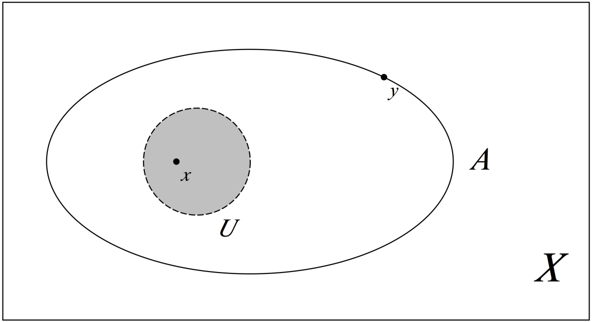

On this view, the interior of A is the set of points where A is "measurably" or "observably" true—that is, where A could *come to be known*.

This notion is then captured by the corresponding modality.

Knowability

By definition, $\Box\varphi$ holds at exactly those worlds where there exists some open witness $U \in \mathcal{T}$ (intuitively, a piece of evidence) that entails φ: $x \in U \subseteq [\![\varphi]\!]$.

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  - Whatever is knowable you can verify is knowable.
  - (All evidence is evidence that it itself exists.)
- ✗ $\neg\Box\varphi \to \Box\neg\Box\varphi$
  - Something may be unknowable without there being any way to verify that it is unknowable.

Unfalsifiability

Recall that we write $\Diamond$ for $\neg\Box\neg$, the dual of $\Box$; we therefore have:

$$\begin{aligned}
[\![\Diamond\varphi]\!] &= [\![\neg\Box\neg\varphi]\!] \\
&= X \setminus int(X \setminus [\![\varphi]\!]) \\
&= cl([\![\varphi]\!]) \\
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\end{aligned}$$

We might then read $\Diamond\varphi$ as "φ is unfalsifiable".

- If $\Diamond\varphi$ is true at state x, then no measurement one could take at x would rule out the possibility of φ.

Topology and accessibility

It is illuminating to consider the connection between topological models and the relational epistemic models considered previously.

- The class of reflexive and transitive frames also corresponds to the S4 axiom system.
  - Specifically: validating $\Box\varphi \rightarrow \varphi$ and $\Box\varphi \rightarrow \Box\Box\varphi$.

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  - Specifically: validating $\Box\varphi \to \varphi$ and $\Box\varphi \to \Box\Box\varphi$.
- Are reflexive and transitive frames "like" topological spaces in some way?

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Theorem

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Proof (sketch).

Define the topology $\mathcal{T}_R$ by insisting that every set of the form $R(x)$ be open. Reflexivity and transitivity guarantee that for each x, $R(x)$ is the *smallest* open set containing x. This simplifies the interior—quantifying over opens is replaced by simply choosing the smallest possible open:

$$\begin{aligned} x \in int([\![\varphi]\!]) \quad &\text{iff} \quad \exists U \in \mathcal{T}_R(x \in U \subseteq [\![\varphi]\!]) \\ &\text{iff} \quad R(x) \subseteq [\![\varphi]\!] \end{aligned}$$

Thus the topological semantics for $\Box\varphi$ agrees with the relational semantics. □

Topology and accessibility

A topological space where every point is contained in a smallest open set is called an *Alexandroff* space.

- In essence, reflexive and transitive frames can be thought of as Alexandroff spaces in disguise.
- There are lots of spaces that do not have this property (e.g., any Euclidean space $\mathbb{R}^n$).
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Epistemically speaking, if there is a smallest open set U containing x, then what could come to be known at x amounts simply to what would be known if U were learned.

- What is knowable is just what would be known given the best evidence.
- Epistemic accessibility in this context captures those worlds that are compatible with the best evidence.

Topological subset models

What if we want to reason about knowledge *and* knowability?

Consider the language $\mathcal{L}_{K,\Box}$ generated by

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \psi \mid K\varphi \mid \Box\varphi,$$

where $p \in \textsc{prop}$.

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A **topological subset model** is a topological space $(X, \mathcal{T})$ together with a valuation $v : \textsc{prop} \to 2^X$ specifying the worlds where each primitive proposition $p \in \textsc{prop}$ is true.

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- x represents the actual world, as usual.
- U represents the agent's current evidence.
  - Intuitively, this is what their knowledge is based on.
  - The condition $x \in U$ captures factivity.
  - The condition $U \in \mathcal{T}$ corresponds to our interpretation of $\mathcal{T}$ as collecting all the possible pieces of evidence.

Topological subset models

These intuitions are formalized in the following semantic clauses:

$$
\begin{array}{lll}
(x,U) \models p & \text{iff} & x \in v(p) \\
(x,U) \models \neg\varphi & \text{iff} & (x,U) \not\models \varphi \\
(x,U) \models \varphi \wedge \psi & \text{iff} & (x,U) \models \varphi \text{ and } (x,U) \models \psi \\
(x,U) \models K\varphi & \text{iff} & U \subseteq [\![\varphi]\!]^U \\
(x,U) \models \Box\varphi & \text{iff} & x \in int([\![\varphi]\!]^U),
\end{array}
$$

where $[\![\varphi]\!]^U = \{x \in U \ : \ (x,U) \models \varphi\}$.

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Theorem

The language $\mathcal{L}_{K,\Box}$ interpreted as above is axiomatized by

$$\mathsf{EL}_{K,\Box} = \mathsf{S5}_K + \mathsf{S4}_\Box + (K\varphi \to \Box\varphi).$$