# Day 5: The Epistemology of Nondeterminism

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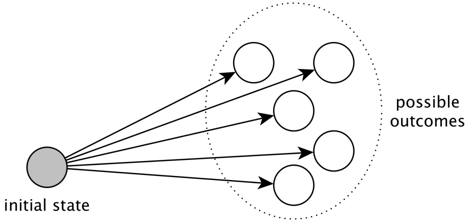
# Propositional dynamic logic

Propositional dynamic logic (PDL) is a framework for reasoning about *nondeterministic program execution*.

Models are relational structures interpreted as state transition diagrams:

- Each program $\pi$ is associated with a binary relation $R_\pi$ on the state space.
- $xR_\pi y$ means that the state $y$ is one possible result of executing $\pi$ in $x$.



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The language of PDL includes a unary modality $\langle \pi \rangle$ for each program $\pi$, where

$$x \models \langle \pi \rangle \varphi \quad \Leftrightarrow \quad \exists y (x R_\pi y \text{ and } y \models \varphi).$$

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Thus, $\langle\pi\rangle\varphi$ is true just in case some possible execution of $\pi$ results in $\varphi$.

What is the sense of *possibility* at play here? We explore an epistemic account.

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1. Reinterpret program execution as fundamentally deterministic.
   - Replace each relation $R_\pi \subseteq X^2$ with a function $f_\pi : X \to X$.
   - Write $\bigcirc_\pi \varphi$ for "after $\pi$, $\varphi$":

$$x \models \bigcirc_\pi \varphi \quad \Leftrightarrow \quad f_\pi(x) \models \varphi.$$

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$$x \models \bigcirc_\pi \varphi \quad \Leftrightarrow \quad f_\pi(x) \models \varphi.$$

2. Enrich the logical setting with a standard knowledge modality $K$, with dual $\hat{K}$, and define

$$\langle \pi \rangle \varphi \ \equiv \ \hat{K} \bigcirc_\pi \varphi.$$

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- But this seems to miss the essence of nondeterminism.
- For a very uninformed agent, we must interpret $\pi$ as having many possible nondeterministic outcomes.
- There is a clear intuitive distinction between those outcomes of $\pi$ that are possible as far as some (possibly quite ignorant) agent knows, and those outcomes that would remain possible *even with good information*.

# Nondeterminism as uncertainty?

Suppose you run a random number generator. This seems a canonical example of a nondeterministic process.

- Not only do you not know what number will be generated, but you are *unable in principle* to determine this in advance.

By contrast, suppose you run a program that prints the next entry in a given database.

- We do not want to call this program nondeterministic, even if you happen to currently be ignorant about the contents of the database.

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By contrast, suppose you run a program that prints the next entry in a given database.

- We do not want to call this program nondeterministic, even if you happen to currently be ignorant about the contents of the database.

This is a distinction we want to respect.

- The relevant epistemic notion is not what any given agent currently happens to know, but what they *could come to know*.

# Topological semantics

We need models rich enough to encode this notion of “potential knowledge” or “knowability”. Of course, we use topology.

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Consider the modal language generated by

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \psi \mid \Box\varphi,$$

where $p \in \textsc{prop}$ and $\Box\varphi$ stands for "$\varphi$ is knowable".

Formulas of this language are interpreted in **topological models** $M = (X, \mathcal{T}, v)$, where:

- $(X, \mathcal{T})$ is a topological space, and
- $v : \textsc{prop} \to 2^X$ is a valuation.

$$\begin{array}{lcl} x \models p & \Leftrightarrow & x \in v(p) \\ x \models \neg\varphi & \Leftrightarrow & x \not\models \varphi \\ x \models \varphi \wedge \psi & \Leftrightarrow & x \models \varphi \text{ and } x \models \psi \\ x \models \Box\varphi & \Leftrightarrow & x \in int([\![\varphi]\!]). \end{array}$$

# Topological semantics

Recall that we write $\Diamond$ for $\neg\Box\neg$, the dual of $\Box$, and correspondingly interpret it as the dual of the interior operator:

$$\begin{aligned} x \models \Diamond\varphi \quad &\Leftrightarrow \quad x \in cl(\llbracket\varphi\rrbracket) \\ &\Leftrightarrow \quad \forall U \in \mathcal{T}(x \in U \text{ implies } U \cap \llbracket\varphi\rrbracket \neq \emptyset). \end{aligned}$$

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We then read $\Diamond\varphi$ as "$\varphi$ is unfalsifiable": no measurement one could take at state $x$ would rule out the possibility of $\varphi$.

# Dynamic topological logic

Topological models equipped with a function $f : X \to X$ have been studied in depth.[1] These are called **dynamic topological models**.

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Of course, we can also equip a topological model with a family of functions $\{f_\pi\}_{\pi \in \Pi}$, and expand the basic language with the dynamic modalities defined previously:

$$x \models \bigcirc_\pi \varphi \quad \Leftrightarrow \quad f_\pi(x) \models \varphi.$$

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# Nondeterminism as unfalsifiability

This setting is rich enough to realize our earlier intuition.

Identify the nondeterministic outcomes of a program $\pi$ with those that cannot be ruled out by *any* feasible measurement:

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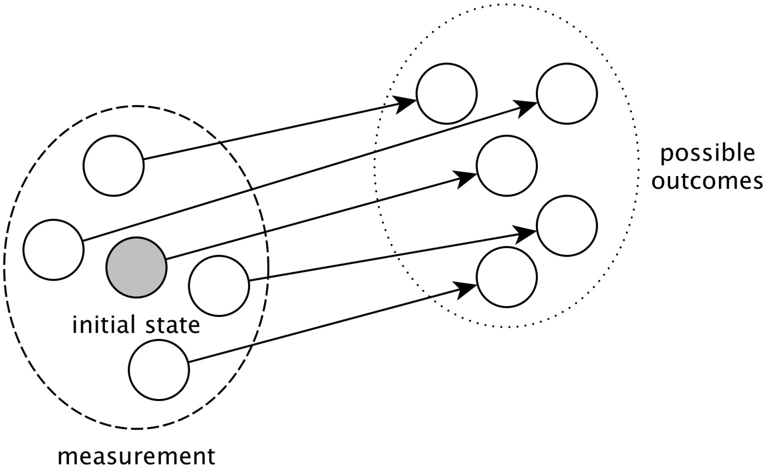
$$\langle \pi \rangle \varphi \;\equiv\; \Diamond \bigcirc_{\pi} \varphi.$$

Dually,

$$[\pi] \varphi \;\equiv\; \Box \bigcirc_{\pi} \varphi.$$

That is, $\varphi$ is a guaranteed outcome of $\pi$ just in case there is some measurement the agent can take *before running* $\pi$ that guarantees $\varphi$ will be true *after running* $\pi$.

# Nondeterminism as unfalsifiability



# Embedding PDL in dynamic topological logic

We will return to refine this understanding of nondeterminism, but first we should check that our topological reinterpretation of PDL is true to the original.

The most basic version of (serial) PDL (without any operations on programs) is axiomatized by

| | |
|---|---|
| CPL | propositional tautologies |
| $\mathsf{K}_\pi$ | $[\pi](\varphi \to \psi) \to ([\pi]\varphi \to [\pi]\psi)$ |
| $\mathsf{D}_\pi$ | $[\pi]\varphi \to \langle\pi\rangle\varphi$ |
| MP | from $\varphi$ and $\varphi \to \psi$ deduce $\psi$ |
| $\mathsf{Nec}_\pi$ | from $\varphi$ deduce $[\pi]\varphi$. |

Call this system $\mathsf{SPDL}_0$.

# Embedding PDL in dynamic topological logic

Of course, we can interpret the language of PDL directly in dynamic topological models via our characterization of $\langle \pi \rangle$:

$$\begin{aligned} x \models \langle \pi \rangle \varphi \quad &\Leftrightarrow \quad \Big( x \models \Diamond \bigcirc_\pi \varphi \Big) \\ &\Leftrightarrow \quad x \in cl(f_\pi^{-1}(\llbracket \varphi \rrbracket)). \end{aligned}$$

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Under these semantics, we obtain the following result.

## Proposition

$\mathsf{SPDL}_0$ *is a sound and complete axiomatization of the language of PDL with respect to the class of all dynamic topological models.*

# Embedding PDL in dynamic topological logic

In fact, given any (serial) PDL model $M = (X, (R_\pi)_{\pi \in \Pi}, v)$, we can construct a dynamic topological model

$$\tilde{M} = (\tilde{X}, \mathfrak{T}, (f_\pi)_{\pi \in \Pi}, \tilde{v})$$

where for every $x \in X$ there exists $\alpha \in \tilde{X}$ such that $x$ and $\alpha$ agree on all formulas of PDL.

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where for every $x \in X$ there exists $\alpha \in \tilde{X}$ such that $x$ and $\alpha$ agree on all formulas of PDL.

- Points $\alpha \in \tilde{X}$ are *networks* of $R_\pi$-paths through $X$:
  - $\alpha : \Pi^* \to X$ where $(\forall \vec{\pi} \in \Pi^*)(\forall \pi \in \Pi)(\alpha(\vec{\pi}) R_\pi \alpha(\vec{\pi}, \pi))$.
- The topology is generated by open sets that group together networks that start at the same point:
  - Basic opens are $U_x = \{\alpha \in \tilde{X} \ : \ \alpha(\emptyset) = x\}$.
- Each $f_\pi$ increments networks by prefixing the program $\pi$:
  - $f_\pi(\alpha)(\vec{\pi}) = \alpha(\pi, \vec{\pi})$.
- Primitive propositions get their values from the network starting point:
  - $\tilde{v}(p) = \{\alpha \in \tilde{X} \ : \ \alpha(\emptyset) \in v(p)\}$.

# Continuity

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- A function $f$ is *continuous* if the preimage of every open set is open: i.e., $f^{-1}(U)$ is open whenever $U$ is open.
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What does continuity correspond to in the present framework?

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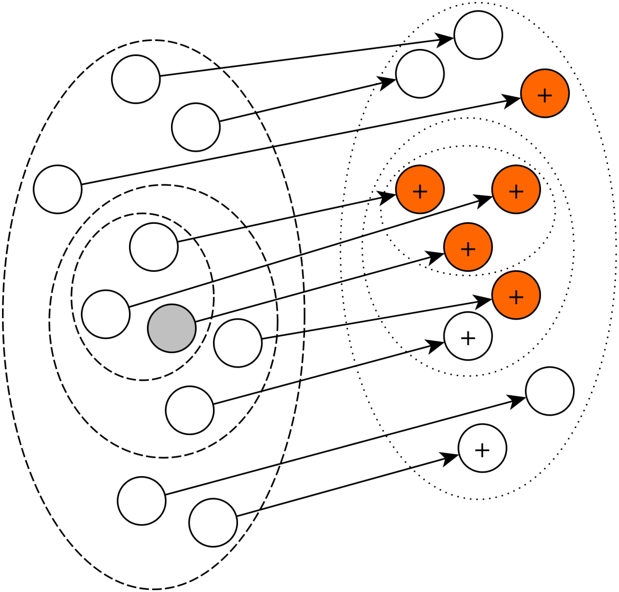
This is determinism! Continuity is determinism.

# Continuity is determinism

This refines our earlier intuitions about (non)determinism.

- It may be that no measurement at $x$ rules out *all* the other states.
  - This is the case whenever $\{x\}$ is not open.
- But this *does not* imply nondeterminism!
- It may still be possible to learn (in advance) everything *that can be known* about the effects of executing $\pi$.
  - This is the case iff $f$ is continuous at $x$.

# Continuity is determinism



# Incorporating knowledge

This interpretation of nondeterminism is epistemic in a certain sense, but involves only “knowability”, not knowledge itself.

What if we want to reason about knowledge as well?

# Topological subset models

*Topological subset models* are well-suited to the simultaneous representation of both knowledge and knowability.

- They underlie previous work on public announcements in topological spaces.
  - The precondition for an announcement of $\varphi$ is taken to be the *knowability* of $\varphi$ (i.e., $\Box\varphi$).
- They have been used to study the interplay between knowledge, knowability, and belief.
  - Stalnaker's principle of "strong belief" (or "full belief"), $B\varphi \to BK\varphi$, is weakened to $B\varphi \to B\Box\varphi$.

# Topological subset models

A **topological subset model** is a topological space $(X, \mathcal{T})$ together with a valuation $v : \text{PROP} \to 2^X$ where truth is defined with respect to *pairs* of the form $(x, U)$ where $x \in U \in \mathcal{T}$.

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$$\begin{array}{lcl}
(x, U) \models p & \Leftrightarrow & x \in v(p) \\
(x, U) \models \neg\varphi & \Leftrightarrow & (x, U) \not\models \varphi \\
(x, U) \models \varphi \wedge \psi & \Leftrightarrow & (x, U) \models \varphi \text{ and } (x, U) \models \psi \\
(x, U) \models K\varphi & \Leftrightarrow & U \subseteq [\![\varphi]\!]^U \\
(x, U) \models \Box\varphi & \Leftrightarrow & x \in int([\![\varphi]\!]^U),
\end{array}$$

where $[\![\varphi]\!]^U = \{x \in U \ : \ (x, U) \models \varphi\}$.

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But we need subset-style semantics for the dynamic modalities:

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Perhaps the most natural definition sets the updated information state to be $f_\pi(U)$.

- This only works if $f_\pi(U)$ is open.

$$(x, U) \models \bigcirc_\pi \varphi \quad \Leftrightarrow \quad (f_\pi(x), f_\pi(U)) \models \varphi.$$

# Dynamic topological subset models

This logic is axiomatized by combining the S5 system for $K$ and the S4 system for $\Box$ with the following:

$$
\begin{array}{rl}
\text{KI} & K\varphi \to \Box\varphi \\
\neg\text{-C}_\pi & \bigcirc_\pi \neg\varphi \leftrightarrow \neg\bigcirc_\pi \varphi \\
\wedge\text{-C}_\pi & \bigcirc_\pi(\varphi \wedge \psi) \leftrightarrow (\bigcirc_\pi \varphi \wedge \bigcirc_\pi \psi) \\
K\text{-C}_\pi & \bigcirc_\pi K\varphi \leftrightarrow K\bigcirc_\pi \varphi \\
\text{O}_\pi & \Box\bigcirc_\pi \varphi \to \bigcirc_\pi \Box\varphi \\
\text{Nec}_\pi & \text{from } \varphi \text{ deduce } \bigcirc_\pi \varphi.
\end{array}
$$

# Dynamic topological subset models

When the functions $f_\pi$ are allowed to be partial, the extra axioms have to be adjusted:

KI $K\varphi \to \Box\varphi$

$\neg$-PC$_\pi$ $\bigcirc_\pi \neg\varphi \leftrightarrow (\neg\bigcirc_\pi\varphi \wedge \bigcirc_\pi\top)$

$\wedge$-C$_\pi$ $\bigcirc_\pi(\varphi \wedge \psi) \leftrightarrow (\bigcirc_\pi\varphi \wedge \bigcirc_\pi\psi)$

$K$-PC$_\pi$ $\bigcirc_\pi\top \to (\bigcirc_\pi K\varphi \leftrightarrow K(\bigcirc_\pi\top \to \bigcirc_\pi\varphi))$

O$_\pi$ $(\Box\neg\bigcirc_\pi\varphi \wedge \bigcirc_\pi\top) \to \bigcirc_\pi\Box\neg\varphi$

Mon$_\pi$ from $\varphi \to \psi$ deduce $\bigcirc_\pi\varphi \to \bigcirc_\pi\psi$.

# Learning?

In a sense, this setting cannot capture *learning*.

- True, an agent's state of knowledge changes in accordance with program execution...
- ...but every "live" possibility $y \in U$ is preserved as the corresponding state $f_\pi(y)$ in the updated information set $f_\pi(U)$.
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Perhaps we should incorporate some tools from *dynamic epistemic logic* (e.g., announcements) to represent acts of information update that truly eliminate states?

In fact, PDL already provides some tools we can adapt for this purpose.

# Test programs

The standard language of PDL is sometimes extended to include "test programs", written $\varphi?$, where $\varphi$ is a formula in the language.

The corresponding relation is defined by

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So the test $\varphi?$ does nothing if $\varphi$ is true and crashes otherwise.

This process is already deterministic; for primitive propositions $p$, we can import $R_{p?}$ as a *partial* function:

$$f_{p?}(x) = \begin{cases} x & \text{if } x \in [\![p]\!] \\ \text{undefined} & \text{otherwise.} \end{cases}$$

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So we have:

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This coincides exactly with the topological definition of a public announcement of $p$.

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- Can we augment this framework to provide a formal epistemic theory of *probabilisitic nondeterminism*, i.e., chance?
  - Represent chance as neither an external feature of the world, nor an internal, subjective feature of an agent...
  - ...but rather as a relationship between the world and the agent's ability to gather information.

Thank you!

# Operations on programs

PDL is often enhanced by equipping $\Pi$ with certain operations:

- sequencing: $\pi_1 ; \pi_2$ executes $\pi_1$ followed immediately by $\pi_2$;
- nondeterministic union: $\pi_1 \cup \pi_2$ nondeterministically executes either $\pi_1$ or $\pi_2$;
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- iteration: $\pi^*$ repeatedly executes $\pi$ some nondeterministic finite number of times.

The latter two may be difficult to interpret in a setting where program execution is fundamentally deterministic.

However, sequencing appears straightforward. Define

$$f_{\pi_1;\pi_2} = f_{\pi_2} \circ f_{\pi_1}.$$

# Topological dynamics with sequencing

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$$\mathsf{SPDL}_0 + \langle \pi_1 ; \pi_2 \rangle \varphi \leftrightarrow \langle \pi_1 \rangle \langle \pi_2 \rangle \varphi.$$

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whereas

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The extra closure operator means we have

$$[\![ \langle \pi_1 ; \pi_2 \rangle \varphi ]\!] \subseteq [\![ \langle \pi_1 \rangle \langle \pi_2 \rangle \varphi ]\!]$$

but not, in general, equality.

# Topological dynamics with sequencing

A function $f$ is called *open* if it maps open sets to open sets: i.e., if $f(U)$ is open whenever $U$ is open.

It is not hard to see that if $f_{\pi_1}$ is open, then

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So the sequencing axiom is valid on the class of dynamic topological models where each $f_\pi$ is open.

- Roughly speaking, $f_\pi$ being open has a "perfect recall" type flavour: what is knowable in advance of executing $\pi$ is also knowable after executing $\pi$.